EFFICIENT RISK MANAGEMENT WITH MONTE CARLO

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Outline:

I - Speeding up Monte Carlo:

- Monte Carlo and Statistical Uncertainties
- Least Squares Importance Sampling (LSIS) and stratification (LSIS+)
    - Examples: European/Asian Options
    - Application to the Libor Market Model

II - Speeding up Risk with Monte Carlo:

- Likelihood Ratio Method
    - Copula-based models
    - Variance Reduction techniques
- Adjoint methods

Monte Carlo Sampling

$$V = E_P\left[G(Z)\right] = \int_D dZ\; G(Z)\, P(Z)$$

“Crude” MC:

$$V \simeq \frac{1}{N_p}\sum_{i=1}^{N_p} G(Z_i) \pm \frac{\Sigma}{\sqrt{N_p}} \qquad Z_i \sim P(Z)$$

Statistical Uncertainty

Variance →

$$\Sigma^2 = E_P\left[G(x)^2\right] - E_P\left[G(x)\right]^2$$

$$\Sigma^2 \simeq \frac{1}{N_p}\sum_{i=1}^{N_p}\left(G(Z_i) - \bar{V}\right)^2$$

$$\boxed{N_p(\text{Given Stat. Error}) \propto \Sigma^2}$$

Importance Sampling

$$\int_D dZ\; G(Z)\, P(Z) = \int_D dZ\; \frac{G(Z)P(Z)}{\tilde{P}(Z)}\, \tilde{P}(Z)$$

Simple Identity

$$V \simeq \tilde{V} = \frac{1}{N_p} \sum_{i=1}^{N_p} W(Z_i)\, G(Z_i) \qquad Z_i \sim \tilde{P}(Z)$$

Sampling Distribution

$$W(Z) = P(Z)/\tilde{P}(Z)$$

Likelihood Ratio

Variance →

$$\tilde{\Sigma}^2 = \int_D dZ\; \left(W(Z)\, G(Z) - V\right)^2\, \tilde{P}(Z)$$

IS: Choose the new probability density in order to decrease Variance

Zero Variance Property

$$P_{\text{opt}}(Z) = \frac{1}{V} G(Z) P(Z)$$

$$W(Z) = \frac{P(Z)}{\tilde{P}(Z)} = \frac{V}{G(Z)}$$

$$\tilde{V} \simeq \frac{1}{N_p} \sum_{i=1}^{N_p} W(Z_i) G(Z_i) = \frac{1}{N_p} \sum_{i=1}^{N_p} V$$

Zero Variance!

Too bad I don't know *V* …..

But I can still try to find a Sampling Distribution that is as close as possible to the Optimal one

Trial Sampling Densities

$$\tilde{P}_\theta(Z)$$

Set of Optimization Parameters

Optimization Problem:

$$\tilde{\Sigma}^2 = \int_D dZ \ \left(W(Z)\,G(Z) - V\right)^2 \, \tilde{P}(Z)$$

Original Measure

$$\tilde{\Sigma}^2_\theta = E_P\left[W_\theta(Z) G^2(Z)\right] - E_P\left[G(Z)\right]^2$$

Least Squares Importance Sampling (LSIS)

Minimize the Variance

$$\tilde{\Sigma}_\theta^2 = E_P\left[W_\theta(Z)G^2(Z)\right] - E_P\left[G(Z)\right]^2$$

… or equivalently minimize:

$$S_2(\theta) = E_P\left[\left(W_\theta(Z)^{1/2}G(Z) - V_T\right)^2\right]$$

Guess for *V*

with Monte Carlo estimator:

$$\simeq \frac{1}{N_p'}\sum_{i=1}^{N_p'}\left(W_\theta(Z_i)^{1/2}G(Z_i) - V_T\right)^2$$

$$\sum_{i=1}^{N_p'}\left(y_i - f_\theta(x_i)\right)^2$$

$$x_i \longrightarrow Z_i$$

$$y_i \longrightarrow V_T$$

$$f_\theta(x_i) \longrightarrow W_\theta(Z_i)^{1/2}G(Z_i)$$

A Least Squares Problem!

Least Squares Importance Sampling (LSIS)

Algorithm:

1) Choose a trial probability distribution and an initial value of the parameters θ

2) Generate a suitable number N_p' of replications of the random variables Z_i

3) Set: $x_i \rightarrow Z_i \quad y_i \rightarrow V_T$

$$f_\theta(x_i) \rightarrow W_\theta(Z_i)^{1/2} G(Z_i)$$

4) Feed the pairs (x_i, y_i) into a non linear Least Square Fitter (e.g., Levenberg-Marquardt) to determine the optimal θ .

Least Squares Importance Sampling

Correlated Sampling makes the approach practical

$$S_2(\theta) \simeq \frac{1}{N_p'} \sum_{i=1}^{N_p'} \left(W_\theta(Z_i)^{1/2} G(Z_i) - V_T \right)^2$$

A limited number of paths is necessary to determine the optimal θ

In fact, the configurations Z_i are fixed. So, the difference between

$S_2(\theta)$ and $S_2(\theta')$

is much more accurate than the Monte Carlo estimate of each of them.

European Call

$$G(Z) = e^{-rT}\left(X_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right] - K\right)^+$$

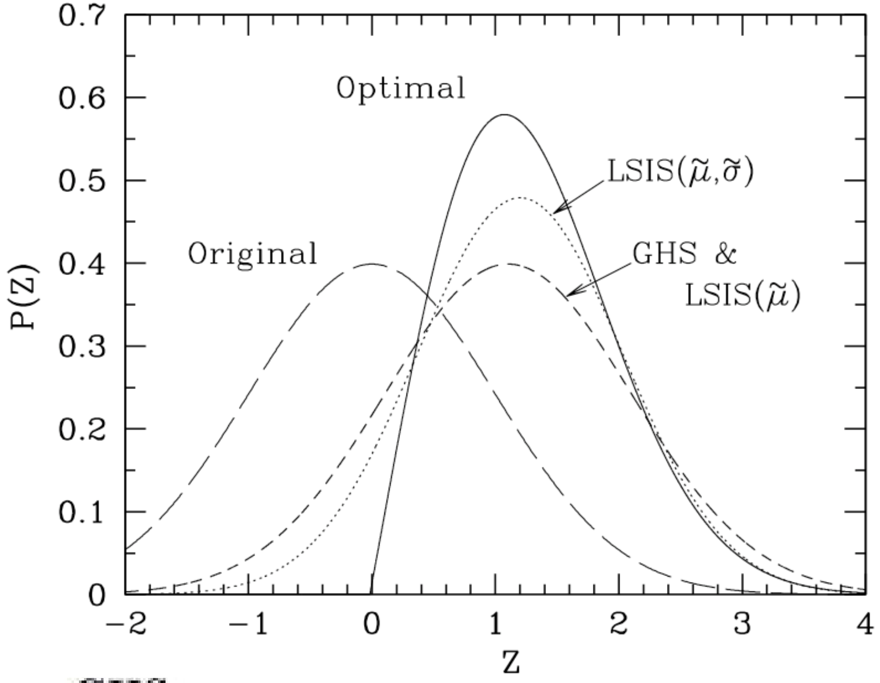
$$P(Z) = (2\pi)^{-1/2}\exp\left(-Z^2/2\right)$$

Trial Density

$$\tilde{P}_{\tilde{\mu}}(Z) = (2\pi)^{-d/2}\; e^{-(Z-\tilde{\mu})^2/2}$$

$$\tilde{P}_{\tilde{\mu},\tilde{\sigma}}(Z) = (2\pi\tilde{\sigma}^2)^{-1/2} e^{-(Z-\tilde{\mu})^2/2\tilde{\sigma}^2}$$

$$N_p' \simeq 50$$



σ	K	LSIS($\tilde{\mu}$)	LSIS($\tilde{\mu}, \tilde{\sigma}$)	RM	GHS
0.1	30	104(1)	1700(100)	112(4)	100(1)
	50	7.8(1)	15(1)	7.8(4)	7.8(1)
	60	33.5(5)	84(5)	31(2)	33.5(5)
0.3	30	16.4(1)	51(1)	16.8(4)	14.8(2)
	50	9.9(5)	27(1)	11(2)	9.9(1)
	60	15.6(1)	35(1)	15.2(4)	14.2(1)

Variance Reduction

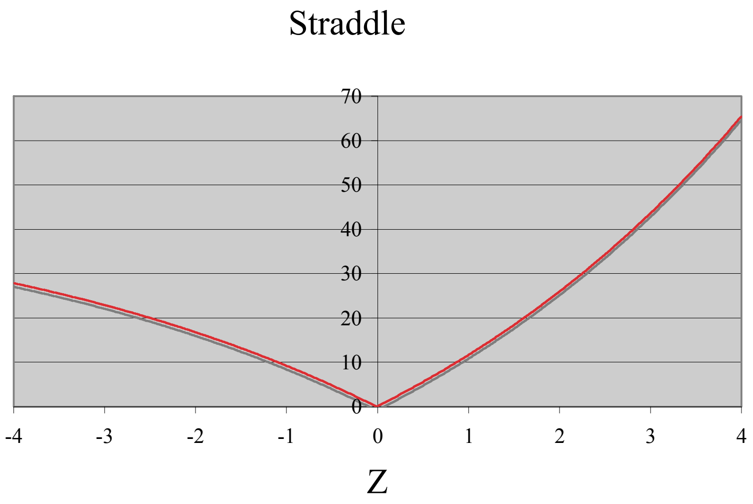
$$\mathrm{VR} = \left(\frac{\sigma(\text{Crude MC})}{\sigma(\text{IS})}\right)^2$$

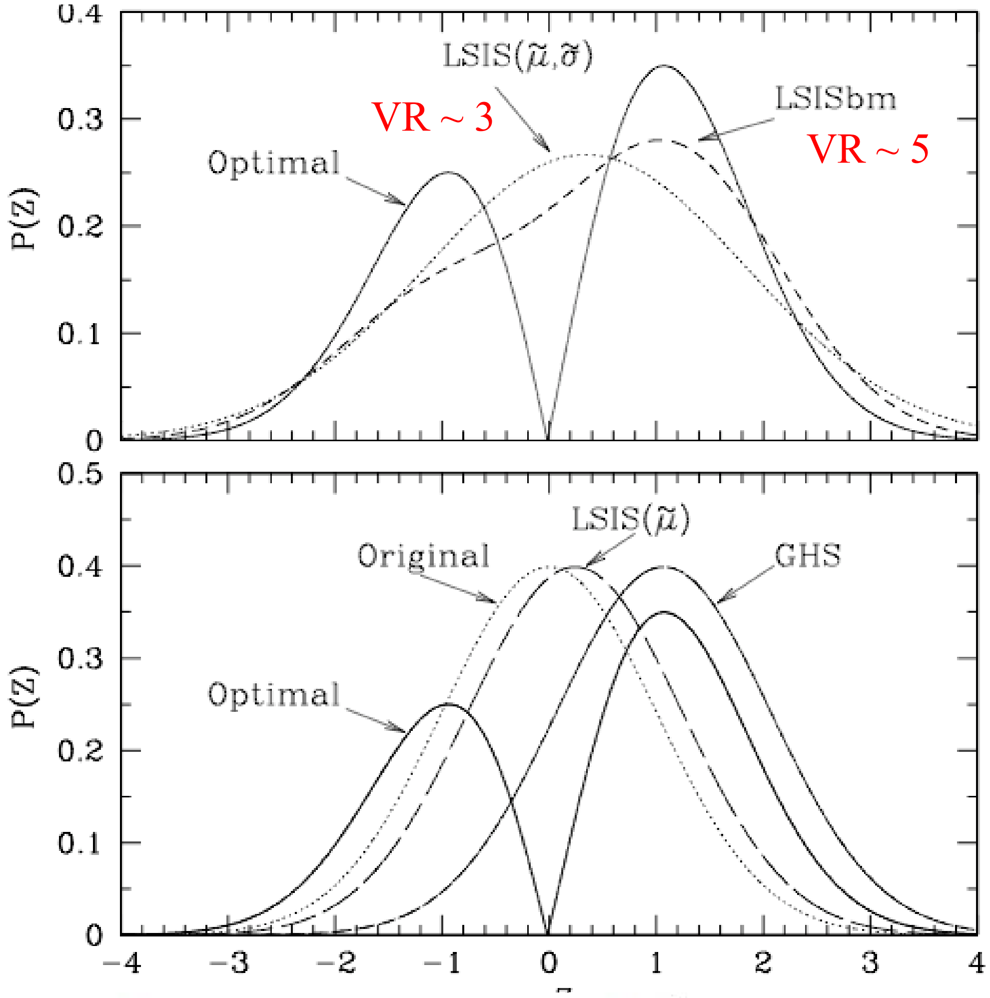
B. Arouna, J. Comp. Finance (2003).

P. Glasserman et al., Math. Finance (1999).

European Straddle

$$G(Z) = e^{-rT}\left|X_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right] - K\right|$$



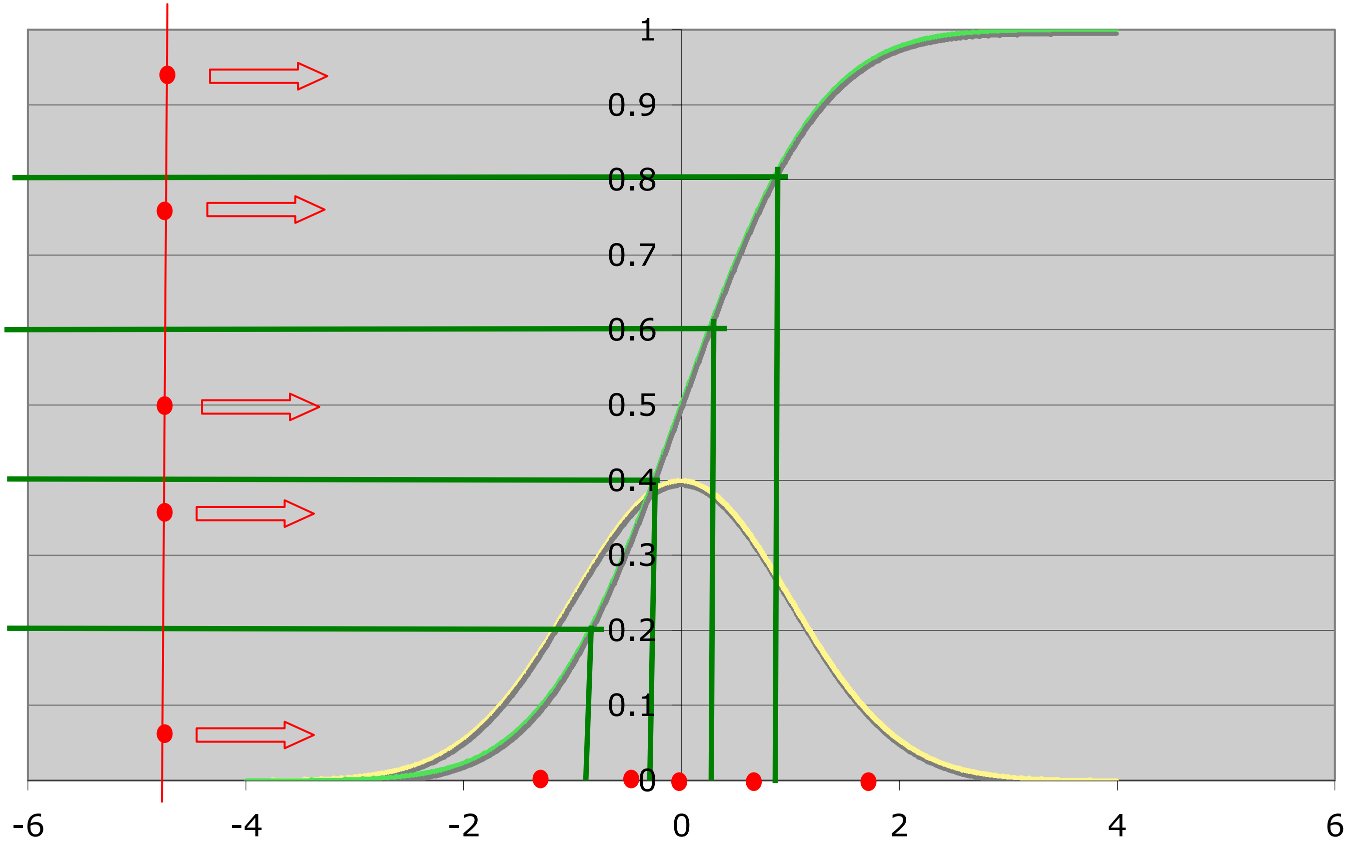


Bimodal Ansatz:

$$\tilde{P}(Z) = (2\pi)^{-d/2}\left[w_a\, e^{-(Z-\mu_a)^2/2} + w_b\, e^{-(Z-\mu_b)^2/2}\right] \quad w_a + w_b = 1$$

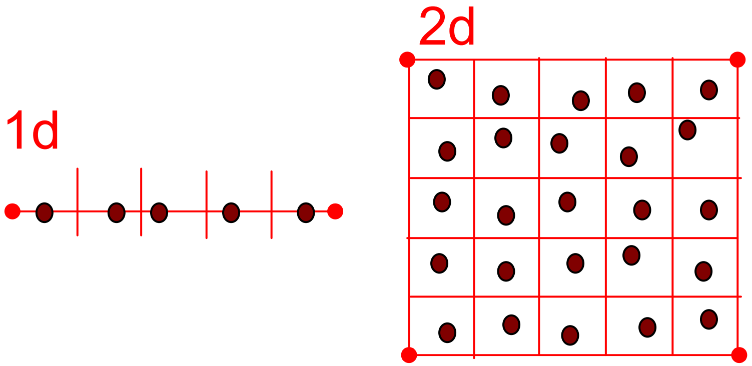
Stratified Sampling

Stratifying a Normal Random Variable



Reducing the Variance by Sampling in a more regular pattern

LSIS + Stratified Sampling (LSIS+)



Too many sample points to fill the space in high dimension *d* !

I can stratify one-dimensional projections!

P. Glasserman et al., Math. Finance (1999).

$$Z^{(i)} = \xi X^{(i)} + (I_d - \xi\xi^t)Y^{(i)}$$

1-d Stratified Normal

$N(0, I_d)$

$\xi \propto \mu(\mathrm{LSIS})$ → LSIS +

Asian Option with Stratified Sampling:

$$G(Z) = e^{-rT}\left(\frac{1}{M}\sum_{i=1}^{M} X_i - K\right)^{+}$$

M	σ	K	VR (LSIS)	VR (LSIS+)
16	0.3	45	8.8(1)	950(20)
		50	9.9(1)	1225(15)
		55	13.6(7)	1900(100)
64	0.3	45	9.0(2)	1060(30)
		50	10.3(5)	1290(30)
		55	12.5(5)	1320(100)

Computational Speed-Up of 3 orders of magnitude!

Libor Market Model Setting

Euler Discretization:

$$\frac{L_i(n+1)}{L_i(n)} = \exp\left[\left(\mu_i(L(n)) - ||\sigma_i(n)||^2/2\right) h_e + \sigma_i^T(n) Z(n+1)\sqrt{h_e}\right]$$

$$\mu_i(L(t)) = \sum_{j=\eta(t)}^{i} \frac{\sigma_i^T \sigma_j h L_j(t)}{1 + h L_j(t)}$$

Risk-Neutral Drift

This fits in the general framework:

$$V = E_P\left[G(Z)\right] = \int_D dZ\ G(Z)\, P(Z)$$

$$P(Z) = N(0, I_d) \equiv (2\pi)^{-d/2}\ e^{-Z^2/2}$$

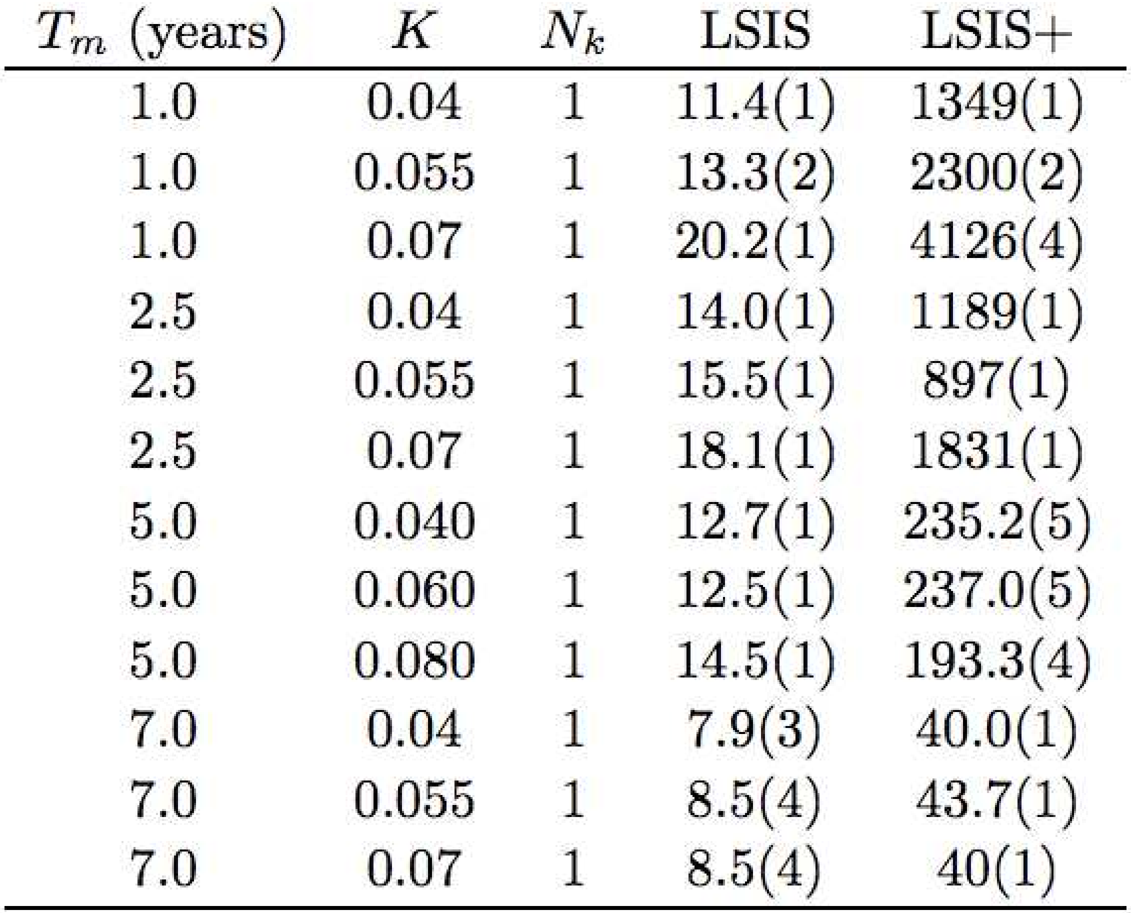
Trial Density

$$\tilde{P}_{\tilde{\mu}}(Z) = (2\pi)^{-d/2}\ e^{-(Z-\tilde{\mu})^2/2}$$

Linear parametrization of the drift (knot points)

Caplet

$$C_h(T_m) = \left(\prod_{i=0}^{m} \frac{1}{1 + hL_i(T_i)} \right) h(L_m(T_m) - K)^+$$



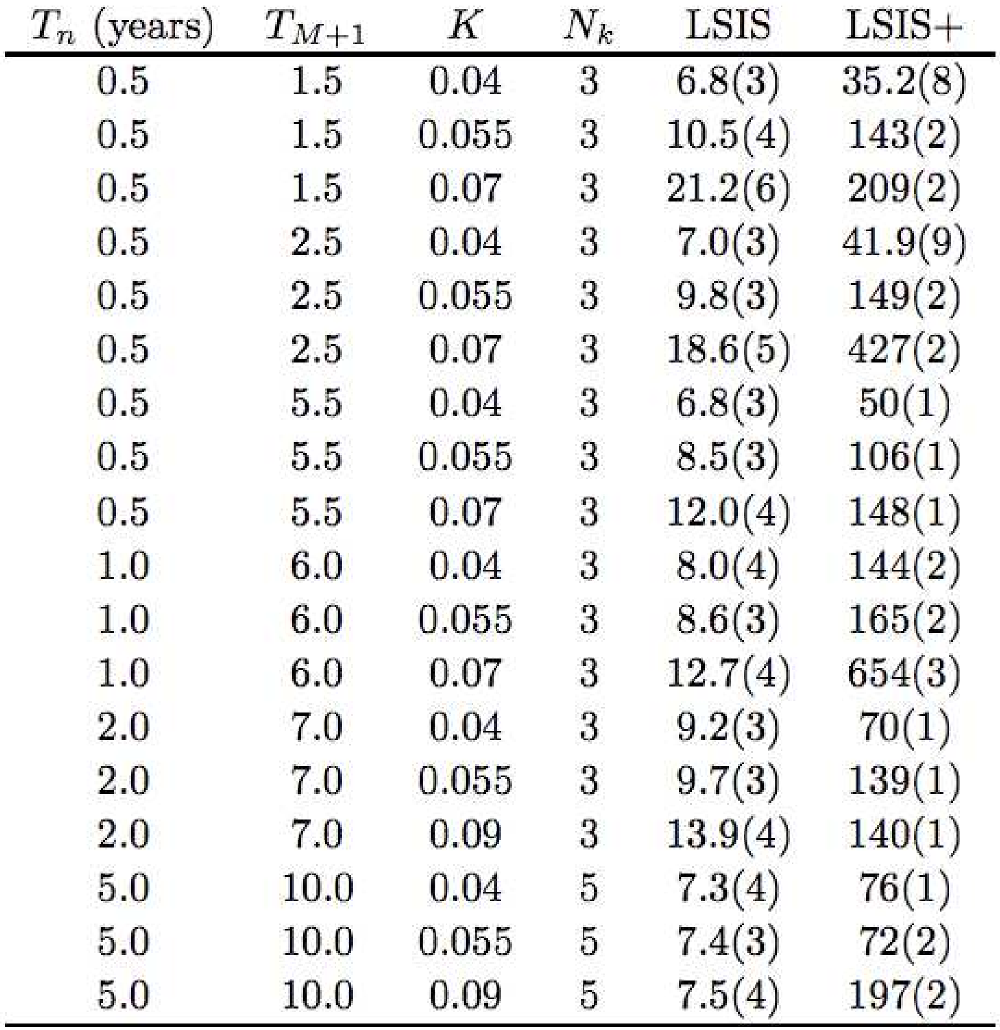
T_m (years)	K	N_k	LSIS	LSIS+
1.0	0.04	1	11.4(1)	1349(1)
1.0	0.055	1	13.3(2)	2300(2)
1.0	0.07	1	20.2(1)	4126(4)
2.5	0.04	1	14.0(1)	1189(1)
2.5	0.055	1	15.5(1)	897(1)
2.5	0.07	1	18.1(1)	1831(1)
5.0	0.040	1	12.7(1)	235.2(5)
5.0	0.060	1	12.5(1)	237.0(5)
5.0	0.080	1	14.5(1)	193.3(4)
7.0	0.04	1	7.9(3)	40.0(1)
7.0	0.055	1	8.5(4)	43.7(1)
7.0	0.07	1	8.5(4)	40(1)

Speed-ups: 2 - 3 orders of Magnitude

$N'_p \simeq 100$

Swaptions

$$V(T_n) = \sum_{i=n+1}^{M+1} B(T_n, T_i) h (S_n(T_n) - K)^+$$



T_n (years)	T_{M+1}	K	N_k	LSIS	LSIS+
0.5	1.5	0.04	3	6.8(3)	35.2(8)
0.5	1.5	0.055	3	10.5(4)	143(2)
0.5	1.5	0.07	3	21.2(6)	209(2)
0.5	2.5	0.04	3	7.0(3)	41.9(9)
0.5	2.5	0.055	3	9.8(3)	149(2)
0.5	2.5	0.07	3	18.6(5)	427(2)
0.5	5.5	0.04	3	6.8(3)	50(1)
0.5	5.5	0.055	3	8.5(3)	106(1)
0.5	5.5	0.07	3	12.0(4)	148(1)
1.0	6.0	0.04	3	8.0(4)	144(2)
1.0	6.0	0.055	3	8.6(3)	165(2)
1.0	6.0	0.07	3	12.7(4)	654(3)
2.0	7.0	0.04	3	9.2(3)	70(1)
2.0	7.0	0.055	3	9.7(3)	139(1)
2.0	7.0	0.09	3	13.9(4)	140(1)
5.0	10.0	0.04	5	7.3(4)	76(1)
5.0	10.0	0.055	5	7.4(3)	72(2)
5.0	10.0	0.09	5	7.5(4)	197(2)

Speed-ups: 1 - 2
orders of Magnitude

Straddle

$$St_h(T_m) = \left(\prod_{i=0}^{m} \frac{1}{1 + hL_i(T_i)}\right) h|L_m(T_m) - K|$$

T_m (years)	K	N_k	LSIS	LSIS (MM)
1.0	0.04	1	2.0(2)	11.6(5)
1.0	0.05	1	1.4(1)	6.4(3)
1.0	0.055	1	1.1(1)	6.1(3)
1.0	0.06	1	1.0(1)	4.0(2)
1.0	0.07	1	1.1(1)	2.6(1)
5.0	0.04	1	6.8(4)	24.6(8)
5.0	0.05	1	5.1(3)	21.2(7)
5.0	0.055	1	4.0(3)	14.8(5)
5.0	0.06	1	3.5(3)	15.5(6)
5.0	0.07	1	2.9(2)	15.8(6)
5.0	0.09	1	2.0(2)	10.0(4)

MM guess provides sizable improvements

Speed-ups:
1 Order of Magnitude

Bimodal Ansatz

$$\tilde{P}(Z) = (2\pi)^{-d/2}\left[w_a\, e^{-(Z-\mu_a)^2/2} + w_b\, e^{-(Z-\mu_b)^2/2}\right] \quad w_a + w_b = 1$$

II - Speeding up Risk with Monte Carlo:

- Likelihood Ratio Method

$$V(\theta) = \int dx_1 \ldots dx_N G(x_1, \ldots, x_N) P_\theta(x_1, \ldots, x_N)$$

$$\bar{\theta}_i = \partial_{\theta_i} V(\theta) = \int dx_1 \ldots dx_N G(x) \partial_{\theta_i} P_\theta(x) \times \frac{P_\theta(x)}{P_\theta(x)}$$

$$\Downarrow$$

$$\bar{\theta}_i = E\Big[G(X) \Omega_{\theta_i}(X) \Big] \qquad \Omega_{\theta_i}(X) = \partial_{\theta_i} \log P_\theta(X)$$

Likelihood Ratio Weight

- Calculation is generally Fast
- Does not require regularity condition on the Payoff
- Requires Knowledge of the PDF
- Variance Properties

Gaussian Copula Models

$$F(x) = \Phi_N\left(\Phi^{-1}\left(M_1(y_1)\right), \ldots, \Phi^{-1}\left(M_N(y_N)\right); \Sigma\right)$$

$$P(x) = \phi_N\left(\Phi^{-1}\left(M_1(x_1)\right), \ldots, \Phi^{-1}\left(M_N(x_N)\right); \Sigma\right) \prod_{i=i}^{N} \frac{m_i(x_i)}{\phi(\Phi^{-1}(M_i(x_i)))}$$

$$m_i(x_i) = dM(x_i)/dx_i$$

Market Implied Marginals

$$\Omega_\theta(x) = \sum_{i=1}^{N} \partial_\theta \log m_i(x_i) - Z(x)^T \left(\Sigma^{-1} - I\right) \partial_\theta Z(x)$$

$$Z_i = \Phi^{-1}(M_i(x_i))$$

- Derivatives of the Mariginals can be calculated (at worst) by bumping.

$$\partial_\theta Z_i = \frac{\partial_\theta M_i(x_i)}{\phi(\Phi^{-1}(M_i(x_i)))}$$

Variance Properties

$$\Omega_\theta(x) = \sum_{i=1}^{N} \partial_\theta \log m_i(x_i) - Z(x)^T \left(\Sigma^{-1} - I\right) \partial_\theta Z(x)$$

$$\bar{\theta}_i = E\Big[G(X)\Omega_\theta(X)\Big]$$

- Difficult to say a priori how good or bad will be the Variance of the LRM estimator.
- Forward-related Risks may diverge for small maturities.

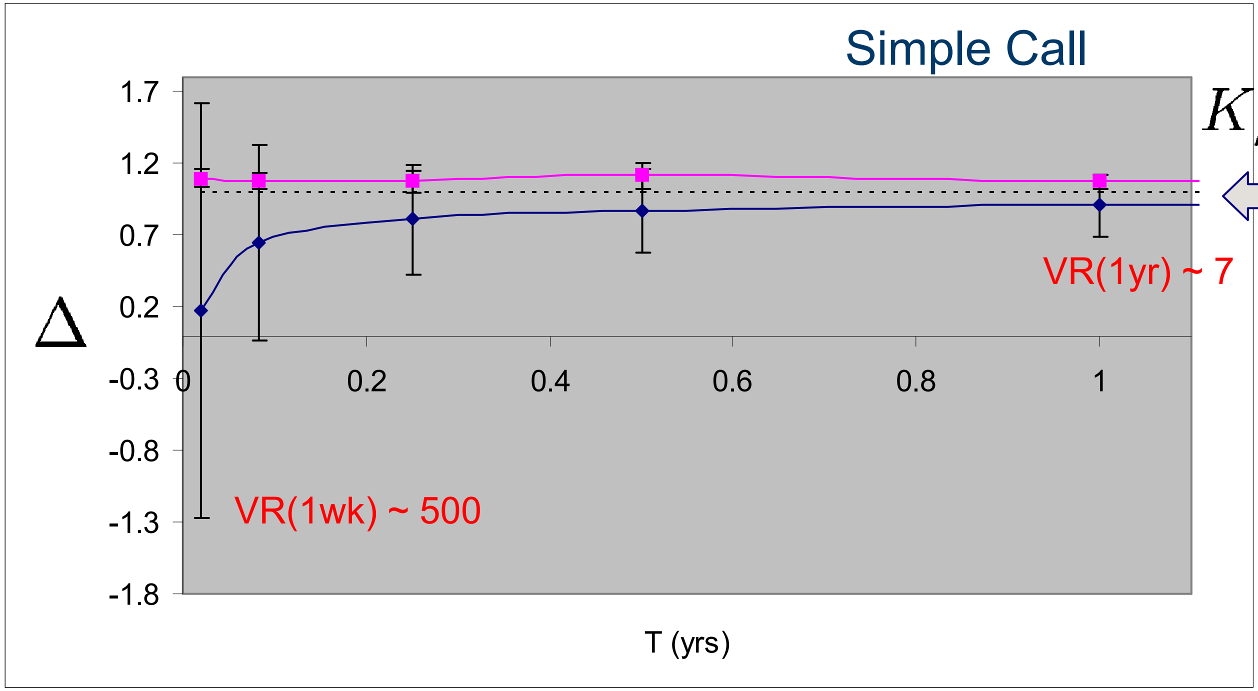
Delta Weight
1d LN Model

$$\Omega = \frac{Z}{X_0 \sigma \sqrt{T}} \quad \Longrightarrow \quad \begin{array}{c} E[\Omega] = 0 \\ \lim_{T \to 0} \mathrm{Var}[\Omega] = \infty \end{array}$$

Variance Reduction Techniques

- Antithetic Variables

$$\Omega = \frac{Z}{X_0\sigma\sqrt{T}} \quad \Rightarrow \quad \Omega = \frac{Z - Z}{2X_0\sigma\sqrt{T}} \equiv 0 \quad \mathrm{Var}[\Omega] = 0$$



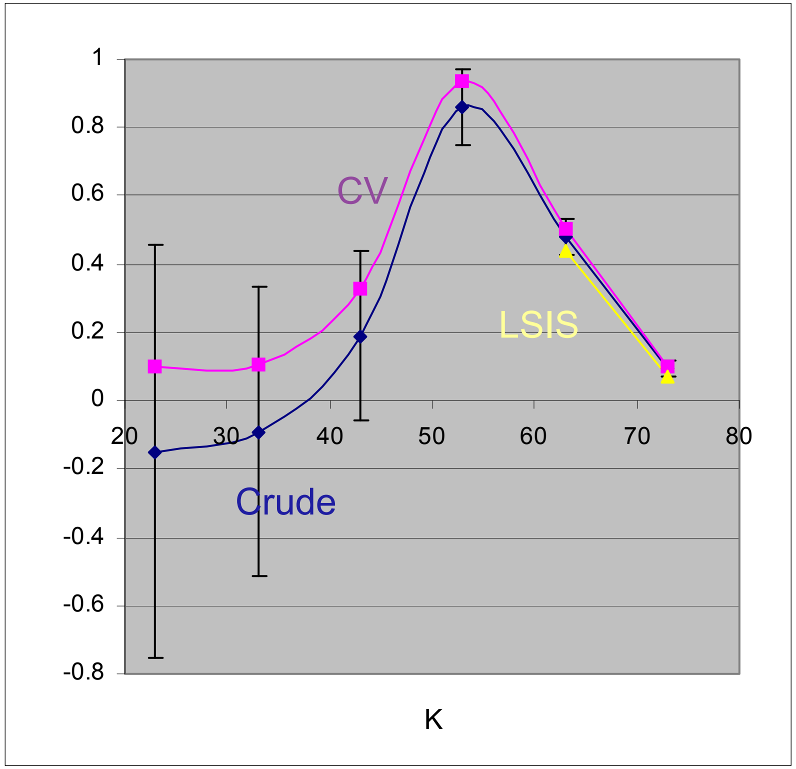
$K/F = 50\%$

Exact

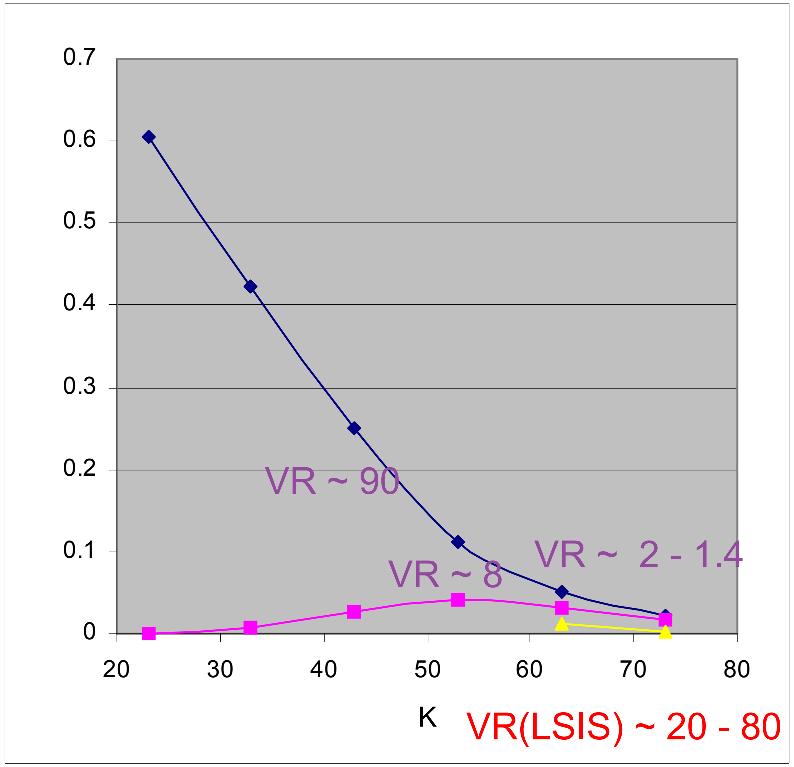
Variance Reduction Techniques

- Control Variates:
  - Weights: $E[\Omega] = 0$
  - Forwards: $\partial_\theta E[X_i] = \partial_\theta F_i$

Vega 6 months 10 Assets Basket Option



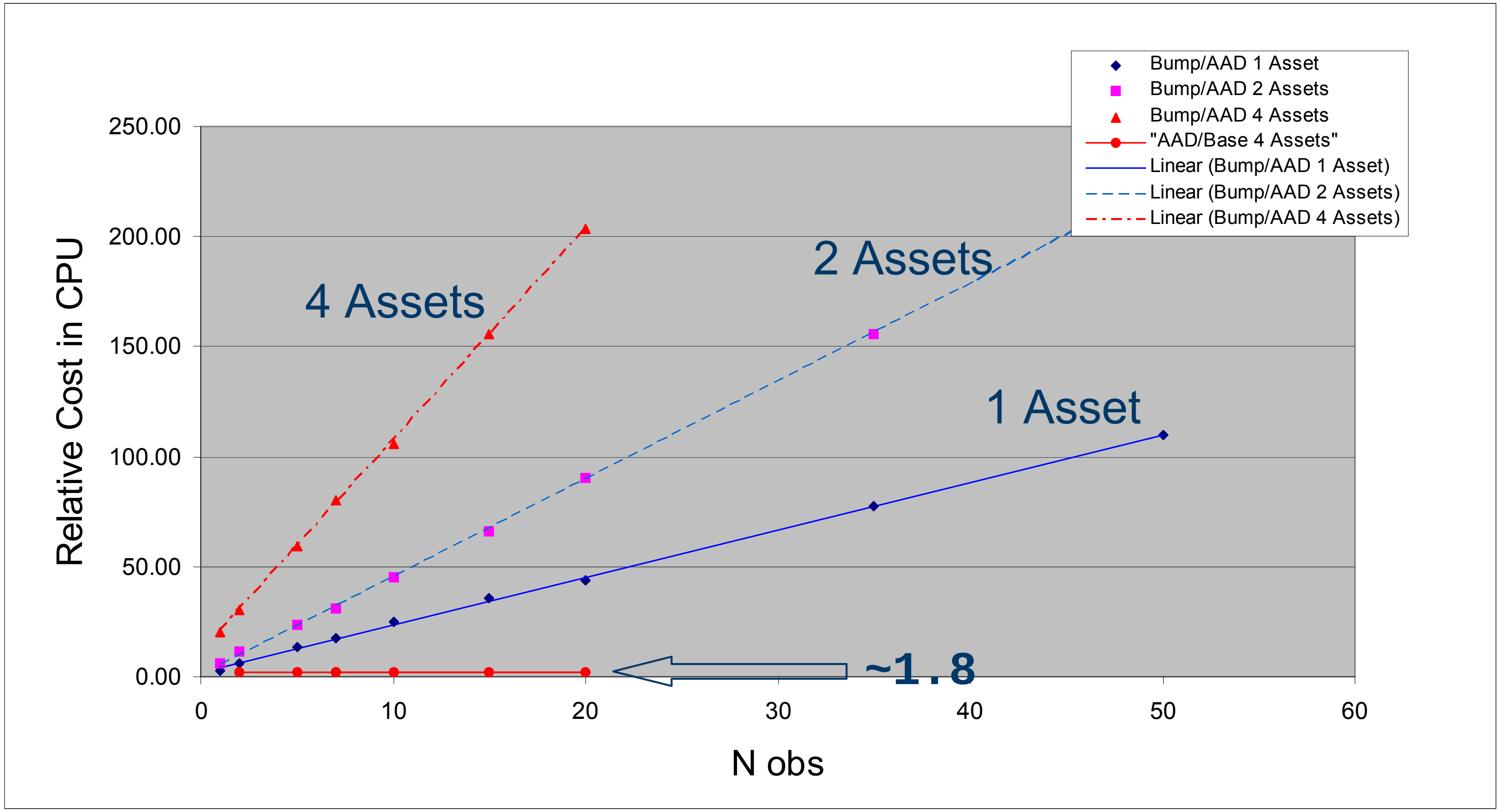
Standard Error



II - Speeding up Risk with Monte Carlo:

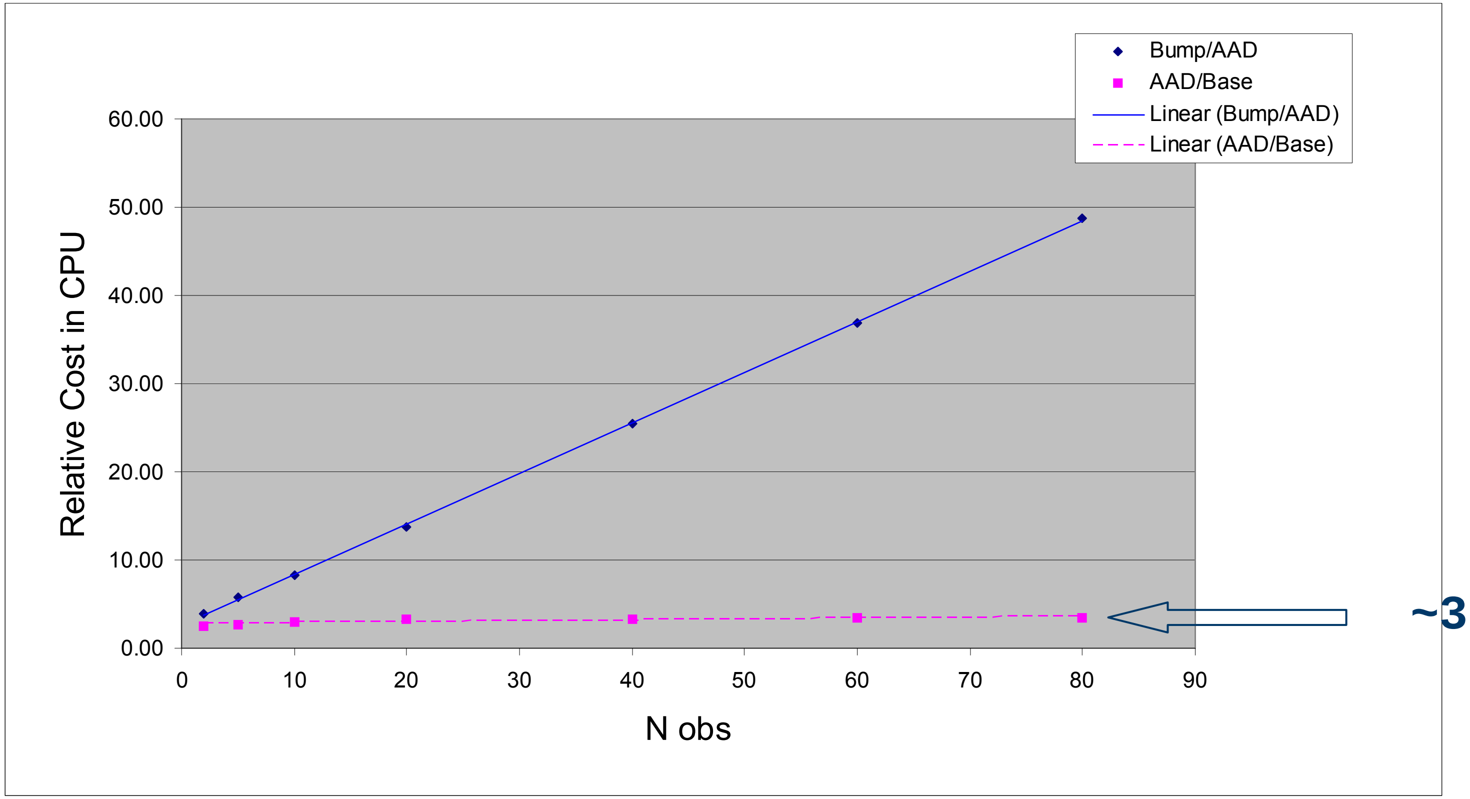
- Adjoint Techniques
  - First introduced in Computational Finance by Giles & Glasserman, in the 2006 `Smoking Adjoints' paper.
  - The approach can be generalized to Path Dependent options under any multifactor model.
  - The variance of the estimators is essentially the same of the naïve bumped counterparts (after smoothing).
  - Remarkable speed-ups especially when a large number of sensitivities is required.
  - Only one drawback: the implementation does not come for free …

- Path Dependent Multi Asset “best-of” style Option



- Risk with respect to the complete term structure of forwards and vols.

- Portfolio of Bond Options under the Hull-White model



~3.4

- Risk with respect to the complete term structure of instantaneous fwd rate, the volatility, and the mean reversion speed.

Summary:

I - Speeding up Monte Carlo:

- LSIS - Least-Squares Importance Sampling:
  - Simple Importance Sampling strategy based on a quick LS Optimization.
  - Can be combined with Stratification for further efficiency gains (LSIS+).
  - LSIS can be used with non-Gaussian/multi-modal trial densities.
  - LSIS and LSIS+ can result in computational savings of orders of magnitude.

II - Speeding up Risk with Monte Carlo:

- Likelihood Ratio Method & Variance Reduction techniques:
  - Antithetic Variables solve the divergence of Delta weights.
  - Simple Control Variates can also help to get stable Risk.
- Adjoint methods:
  - Provide a very efficient and general framework for Risk calculation.

L. Capriotti, Quantitative Finance (in press); Wilmott Magazine, Sep. 2007.